Reconnection of vortex filaments in the complex Ginzburg-Landau equation

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A criterion for the reconnection of vortex filaments in the complex Ginzburg-Landau equation is presented. In particular, we give an estimate of the maximum intervortex separation beyond which coplanar filaments of locally opposite charge will not reconnect. This is done by balancing the motion of the filaments toward each other that would result if they were straight (a two-dimensional effect) with the opposing motion due to the filament curvature. Numerical experiments are in good agreement with the estimated vortex separation.
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Amplitude equations, which describe the slow spatial and temporal variation of a system in the vicinity of a linear instability, have proven to be a fruitful tool for the theoretical analysis of pattern formation in nonequilibrium systems [1,2]. A prototypical amplitude equation is the complex Ginzburg-Landau equation (CGLE) which describes the behavior of extended systems near a Hopf bifurcation of a steady, homogeneous state to a limit cycle [3]. In particular, the CGLE applies to reaction-diffusion systems [4]. The CGLE is given by

$$\frac{\partial A}{\partial t}=A-(1+i\alpha)|A|^2A+(1+i\beta)\nabla^2 A, \tag{1}$$

where α and β are real parameters, and $A(\mathbf{x},t)$ is a complex scalar field often called the order parameter. The CGLE displays a rich variety of behavior [1]. In two dimensions, it possesses spiral wave solutions and is perhaps the simplest equation that does so. Expressed in polar coordinates, the single-armed solution for an isolated two-dimensional (2D) CGLE spiral wave is of the form

$$A_0(r,\theta,t)=F(r)\exp\{i[-\omega_0 t+\sigma\theta+\psi(r)]\}. \tag{2}$$

The symbol $\sigma=\pm 1$ denotes the ''topological charge'' of the spiral wave, as one can see that the phase change of A_0 upon counterclockwise traversal of a path around the origin is $\pm 2\pi$. Hence the center of the spiral is a vortex or ''defect.'' This singularity in the phase field forces A_0 to vanish at the center of the spiral. The real functions $F(r)$ and $\psi(r)$ have the asymptotic behaviors $F\sim r$, $\psi'\sim r$ as $r\to 0$, and $F\to\sqrt{1-k_0^2}$, $\psi'\to k_0$ as $r\to\infty$, with the prime signifying differentiation with respect to r. The amplitude F rises rapidly in the region called the core that immediately surrounds the defect. The frequency ω_0 is determined uniquely by the parameters α and β [5,6]. This, in turn, uniquely specifies the asymptotic wave-number k_0 via the plane wave dispersion relation $\omega_0=\alpha+(\beta-\alpha)k_0^2$.

Spiral waves can also occur in three-dimensional (3D) systems (including the CGLE) and are known as scroll waves [7,8]. The point defect at the center of a 2D spiral wave is replaced by a vortex filament about which the spirals rotate. This filament can span the boundaries of the medium or it can be closed to form a loop, which may be knotted or interlinked with other loops [9]. Experimentally, scroll waves have been observed in excitable media such as the Belousov-Zhabotinsky reaction [10], in slime mold [11], and in electrical signal propagation in the heart [12].

Reconnection of vortex filaments leading to topological changes in the configuration occurs when the filaments come into local contact. For instance, as two interlinked filament loops both contract, they may make local contact and reconnect, creating a single large loop. Experimentally, vortex filament reconnection has been observed in fluids [13,14] and in liquid crystals [15]. Reconnection has been demonstrated numerically in the Navier-Stokes equations [16,17], a coupled map lattice version of Eq. (1) with $\alpha=\beta=0$ [18], and the nonlinear Schrodinger equation [Eq. (1) in the limit $\alpha,\beta\to\infty$] [19]. Winfree and Strogatz [9] showed the reconnection of scroll wave filaments of reaction-diffusion equations to be possible based on topological considerations, and it has been observed in numerical simulation of excitable media [20]. In this paper, we present numerical evidence for the reconnection of scroll filaments in the 3D complex Ginzburg-Landau equation, and estimate the threshold separation between coplanar filaments of given curvatures beyond which reconnection will not take place. The occurrence of reconnection in CGLE vortex filaments has important implications. In particular, reconnection can destroy the topological integrity of linked or knotted filaments. In addition, reconnection will allow an initially dense state of vortices to evolve into a smaller number of larger vortices. Reconnection may also be an important feature of defect-mediated turbulence [21–24] in three dimensions, perhaps leading to the existence of large-scale, long-lived structures.

The reconnection of two initially coplanar scroll wave filaments is shown in Fig. 1 for a numerical simulation of the CGLE in three dimensions, using parameter values of $\alpha=1$ and $\beta=0$. The amplitude field $|A|$ is displayed, and the filaments are seen as dark regions of low $|A|$. In the region where the filaments are initially nearest, they draw closer to one another. Eventually they merge and form a new pair of

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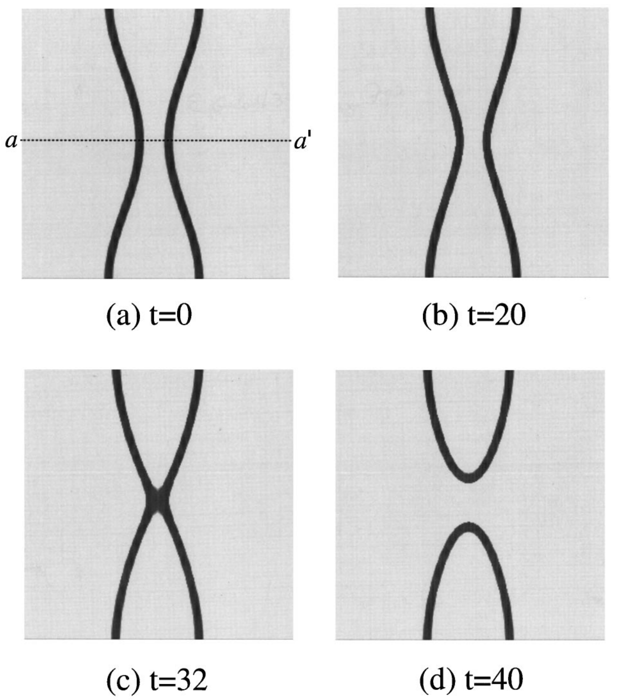

FIG. 1. Amplitude field for reconnecting coplanar filaments. The locus of points in the data volume for which $|A|<0.5$ is shown. As the filaments move toward each other, they also drift together in the direction perpendicular to the plane of the figure. Since this drift is larger where the filaments are closer together, the initial planar disposition of the filament becomes nonplanar. This nonplanarity, however, is a small effect.

filaments which then move apart from each other. The filaments become distorted only in the immediate neighborhood of where they are closest, and so the reconnection process is a fairly localized phenomenon.

Figure 2 shows the phase field of the complex quantity A in the plane perpendicular to the initial plane of the filaments, with the intersection being formed by the line aa' in Fig. 1(a). This plane intersects the filaments at the points where they are nearest (henceforth referred to as the center plane). The singularities in Fig. 2(a) are the centers of the spirals where the different phase bands pictured in the plot converge. The two spirals are seen to be of opposite charge, winding in opposite senses around their centers (the condition of outward-directed group velocity selects a preferred winding direction for a given topological charge). We note that in general a global topological charge cannot be assigned to a spiral wave filament, but the direction, clockwise or counterclockwise, in which the phase increases around a given point on the filament can be used to pick out a direction along the tangent to the filament at that point via the right-hand rule. For instance, a plane that slices through a circular filament ring along its diameter will reveal two spirals of opposite charge, but the right-hand rule defines a unique direction of travel along the ring. Any mention of topological charge when referring to spiral wave filaments is understood to be made in the context of a given two-dimensional plane (the center plane for the case of Fig. 2) and a given point along the filament intersected by that plane. Figure 2(b) shows the phase in the center plane after reconnection, and one can see that the singularities have disappeared. A view of the phase in the plane perpendicular to the center plane and that cuts through the middle of line aa' would reveal the reverse process—the creation of a pair of oppositely charged singularities which then proceed to move away from each other.

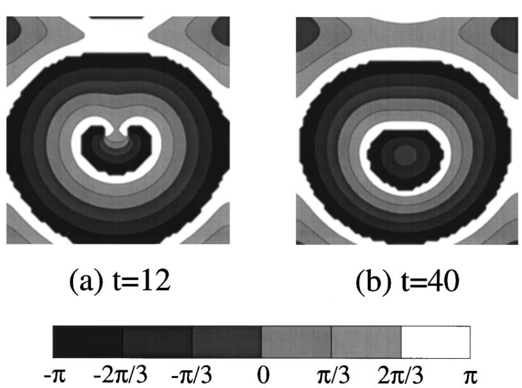

FIG. 2. Phase field (a) before and (b) after reconnection along the center plane.

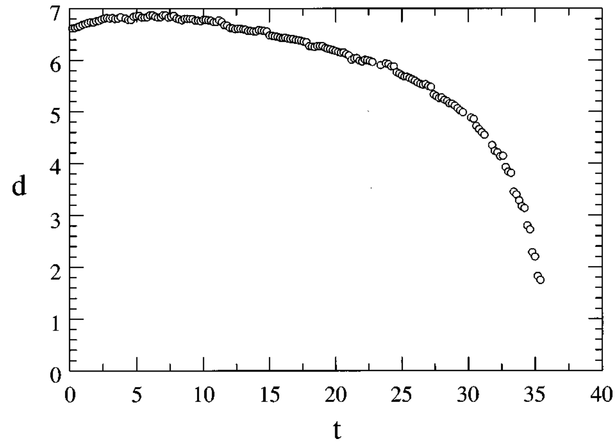

FIG. 3. Intervortex separation in the center plane vs time for reconnecting filaments.

Our numerical solutions of the CGLE were implemented using a pseudospectral code. The size of the computational box is 128^3 grid points, and each side is 20π space units long. The time step between iterations is 0.1. The initial shape of the filaments is a sine wave with an amplitude of $\Gamma=3.54$ space units and a wavelength $L=20\pi$. Considering only one of the filaments, the field around it is initialized, so that A in the plane perpendicular to a given point on the filament approximates the 2D spiral solution for the particular parameters α and β. The field of the filament in the other half of the box is simply the mirror image of the first.

The distance d between the centers of the filaments in the center plane as a function of time is shown in Fig. 3. The filaments are initially separated by 6.6 space units. The filaments approach each other at an increasing rate, and reconnection occurs at about time $t=36$. At a given moment, the positions of the filament vortices in the center plane are determined by finding the two unit grid squares around which the change in phase is $\pm 2\pi$ (it is zero elsewhere). If only this method were used, however, the separation versus time plot would have a staircaselike appearance due to the low grid resolution. There are only 13.3 grid divisions initially separating the filaments, and, furthermore, the filaments move as mirror images of each other, making the jump between grid squares at exactly the same time. This would leave only six ''steps'' on the staircase which is unsatisfactory for the determination of the filament velocity needed below. Therefore, the following interpolation scheme is employed to obtain Fig. 3. The field amplitudes $|A|$ at the four grid points comprising the square containing the singularity are compared, and the lowest one is found. Presumably, this is the grid point closest to the singularity, and the distance between this grid point and the phase singularity at the filament's center is estimated using the field of the core of a 2D vortex with the same α and β, obtained from a numerical simulation with much higher resolution (a 1024^2 grid for the same side length of 20π). The distance between the singularity and another grid point in the square can also be found in this way. These two distances and the distance between adjacent grid points form the legs of a triangle, and so the location of the singularity within the grid square can be determined. This method works fairly well but not perfectly, as can be seen from the choppiness of Fig. 3.

The reconnection process of the coplanar filaments depicted in Fig. 1 can be understood qualitatively as being driven by the essentially 2D attraction between oppositely charged spirals, and being resisted by the 3D effect that seeks to straighten out the curved filaments. The interaction between CGLE spiral wave vortices in two dimensions has received much attention. The nature of the 2D interaction between opposite charges depends on the parameters α and β and the distance d between them. It can be attractive or repulsive. The charges can move toward each other and annihilate, form a bound state (in which d becomes constant), or asymptotically repel each other. The defects have oppositely directed velocities along the axis connecting them, and they can also drift in the direction perpendicular to this axis (these components are parallel). For spirals whose centers are well separated and with very long asymptotic wavelengths ($|\alpha-\beta|\ll 1$) in comparison to d, the strength of the interaction as measured by the defect velocity falls off as $1/d$ [25–27]. When the vortices are still well separated and the asymptotic wavelength is now comparable to d, a ''shock'' (i.e., a ridge of elevated $|A|$) forms where the waves emanating from the vortices collide. Its effect is to screen the interaction between the vortices strongly, and, in this case, the velocity due to interaction decays in an essentially exponential fashion with d [28]. Aranson, Kramer, and Weber demonstrated that, for $(\alpha-\beta)/(1+\alpha\beta)>0.845$, oppositely charged CGLE vortices can form stable bound states where the separation distance remains constant [29]. For small enough d, however, opposite charges attract and eventually annihilate. This is the region of relevance to reconnection.

An asymptotic theory for the dynamics of 3D CGLE vortex filaments in the absence of filament-filament interaction was developed in Ref. [30]. The motion of the filaments is due to the local curvature κ, when $\kappa\ll 1$ (i.e., κ^{-1} is large compared to the vortex core). The velocity v_κ at which the filament moves towards its center of curvature is given by

$$v_\kappa=(1+\beta^2)\kappa. \tag{3}$$

This formula agrees well with numerical simulations of scroll rings up to κ values of about 0.2. There is no drift of planar filaments in the direction normal to their plane to first order in κ. Equation (3) implies that a low curvature, sinusoidal filament will relax exponentially to a straight line. For the configuration of Fig. 1, the filament velocity due to the curvature resists reconnection but is too low to prevent it. After reconnection, however, the high local curvature of the filaments leads to fairly rapid separation.

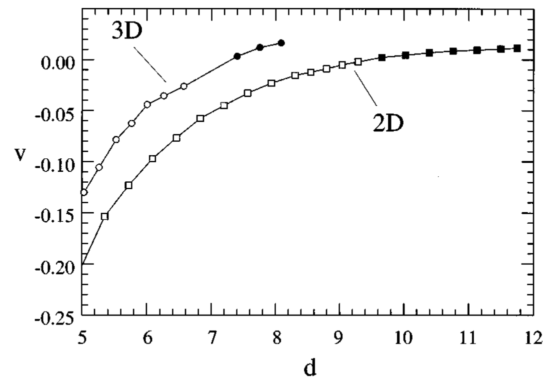

FIG. 4. Velocity vs intervortex separation. The circles are for coplanar filaments and the boxes are for a two-dimensional pair of opposite charges. See text for details.

A prediction of the threshold separation, d_c, beyond which reconnection will not occur for coplanar filaments of a given curvature, can be obtained by balancing the curvature-induced velocity with the velocity of attraction between 2D vortices of opposite charge which are separated by the same distance as the filaments. We obtain the 2D vortex velocity due to mutual interaction directly from numerical simulation of the 2D CGLE, as there is no analytical expression for it. For the parameter values $\alpha=1$ and $\beta=0$, Fig. 4 displays the component of velocity along the axis connecting the defects as a function of their separation (all subsequent references to velocity refer to this component). The lower curve is for a pair of oppositely charged 2D vortices, and the upper curve is for the center plane of the coplanar, sinusoidal scroll filament configuration. Each curve consists of data from two separate numerical runs. For the 2D curve, the negative velocity data set (open squares) starts from a defect separation of $d=9.3$ and the defects attract each other and eventually annihilate. The size of the computational box is 512^2 grid points, with a side length of 20π. The positive velocity data points (solid squares) start at $d=9.6$, and in this run the defects repel each other. The 3D velocity curves are obtained from simulations on a 128^3 box with a side length of 20π. The negative velocity part (open circles) of the 3D velocity curve corresponds to a run where reconnection takes place, and the positive velocity section (solid circles) comes from a run where the filaments move away from each other. The filaments have the same initial curvature in both cases. The relatively large gap between initial data points of the negative and positive sections of the 3D curve ($d=6.6$ and 7.4) is due to the inability to get very close to d_c, given the high computational costs in three dimensions. The 3D velocity curve is bumpy in places due to the combination of low grid resolution and small velocities. Brief initial transients, such as the positive velocity transient seen in Fig. 3, are not included.

Estimating d_c by balancing the 2D vortex velocity v_{2D} with the curvature-induced velocity v_κ is equivalent to the claim that the filament velocity should be approximately given by $v_{2D}+v_{\kappa}$ for sufficiently well-separated filaments of low curvature. This additivity of the two effects is expected to hold to lowest order in a perturbation expansion in v_{2D}, $v_{\kappa}\ll 1$ with a next order correction proportional to the product $v_{\kappa}v_{2D}$. The curvature at the crest of a sine wave of amplitude Γ and wavelength L is $\kappa=(2\pi/L)^2\Gamma$, which for our filaments yields an initial curvature of $\kappa=0.035$. Equation (3) then gives an initial curvature-induced velocity of $v_{\kappa}=0.035$. For the 3D run in which reconnection does not occur (solid circles in Fig. 4), the initial difference between the 3D and 2D velocity curves is 0.041. For the reconnection run (open circles in Fig. 4), the initial difference is 0.044. These values agree reasonably well with the predicted difference of 0.035, given the poor quality of the 3D curve and the neglected departure from the additivity of the 2D attraction and curvature effects.

The estimated threshold distance for reconnection can now be obtained by finding the point at which the velocities v_{2D} and v_{κ} sum to zero. From the 2D velocity curve in Fig. 4, one finds by setting $v_{2D}=-0.035$ that the predicted reconnection threshold is $d_c=7.5$. The measured value of d_c is found from the point at which the 3D velocity curve passes through zero, which gives $d_c=7.3$. This is good agreement, especially considering that the distance between adjacent grid points is 0.49. In the case where the filaments have unequal curvatures κ_1 and κ_2 at the points where they are least separated, d_c is determined from the condition $v_{2D}=(v_{\kappa_1}+v_{\kappa_2})/2$.

The maximum separation for the annihilation of oppositely charged 2D spirals, corresponding to $v_{2D}=0$ on Fig. 4, is 9.5. It is also the threshold of reconnection of coplanar filaments regardless of curvature, since they behave as 2D vortices in the limit of zero curvature. We have observed the reconnection of noncoplanar filaments, and it appears that for straight filaments the reconnection threshold is less than for equivalent coplanar filaments. If this is the case, then the 2D annihilation threshold is likely the upper bound on the filament separation required for reconnection, except in configurations where the curvature-induced velocity actually aids reconnection (as would be true for interlinked loops which would collapse into each other).

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